Program of
Banach Center - Oberwolfach Graduate Seminar: **Geometry and Topology of Compact Homogeneous Spaces**

Bedlewo Conference Center, November 21 - 25, 2022

Organizers:
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1 Short description of the program

The seminar addresses to about 30 graduate students and early stage post-docs who are interested to deepen not only standard results about compact homogeneous spaces but also to learn more specialized topics, e.g. on the classification of such spaces, exotic structures and moduli spaces of metrics with special curvature properties.

A homogeneous space is a smooth manifold M with a transitive operation of a Lie group G. Fix a point $x \in M$, then M is diffeomorphic to the quotient space of left cosets of the stabilizer subgroup H of x (H depends on the choice of x but is unique up to conjugation). The following facts are well-known: If M is compact and simply connected, it is possible to modify G and H such that they become compact, G simply connected and H connected. Thus one gets all simply connected compact homogeneous spaces of dimension n by considering connected closed subgroups H of codimension n in simply connected compact Lie groups G up to conjugation. A priori there are infinitely many of such pairs $H \leq G$ for fixed n but there is a finite algorithm to construct all irreducible pairs [11]. Based on this systematic construction, it is possible to give an explicit diffeomorphism classification up to dimension 6 (see also [7]), and as a "miracle", there are exactly 20 diffeomorphism types in dimension 8 [11]. In contrast to this, the first exotic homogeneous spaces appear in dimension 7 with an infinite number of diffeomorphism types [12], [13] and it seems unrealistic that a manageable diffeomorphism classification of all spaces M beyond dimension 8 could exist. If one restricts to spaces which are not only 1-connected but $1\frac{1}{2}$-connected (i.e. $\pi_2(M)$ is torsion), it is possible to push the classification up to dimension 12 [14]. These results involve not only a lot of Lie theory, representation theory and topology, but also more advanced methods, e.g. homotopy theory, spectral sequences, characteristic classes, bordism and surgery theory.

Compact homogeneous spaces do also play a prominent role in global Riemannian geometry and the theory of non-negative and positive sectional curvature. Indeed, many spaces with these curvature properties arise here from taking an isometric quotient of a compact Lie group equipped with a bi-invariant metric, and in the especially intriguing realm of positive curvature and the quest for new examples there, it took more than half a century and the efforts of Berger, Wallach, Aloff-Wallach, Berard-Bergery, Wilking and Wilking-Ziller to complete the classification of all simply connected, compact positively curved Riemannian homogeneous spaces (see [21] and the further references given there). Moreover, these results also inspired the investigation of corresponding curvature properties of natural generalizations of homogeneous spaces like cohomogeneity one manifolds and biquotients.

Instead of trying to construct, classify or obstruct Riemannian metrics with certain curvature properties, one can also ponder the question what the space of all such metrics on a given manifold M looks like, or ask a similar question for its moduli space, i.e., its quotient by the full diffeomorphism group of M, acting by pulling back metrics. These spaces are customarily equipped with the topology of smooth convergence on compact subsets and the quotient topology, respectively. The topological properties of these objects hence provide the right means to measure "how many" different metrics and geometries the manifold M does exhibit, and since H. Weyl's early result from 1916 on the connectedness of the space of positive Gaussian curvature metrics on S^2 and the foundings of Teichmüller, infinite-dimensional manifold and Lie group theory, uniformization and geometrization, the study of spaces of metrics and their moduli has been a topic of interest for differential geometers, global and geometric analysts and topologists alike [19]. Compact homogeneous spaces have in these studies also been of special importance (compare, e.g., [5], [20]).

2 Schedule

We will give 4 lectures of 45 minutes in the morning with breaks of 10 minutes according to the schedule 9-9:45 am, 9:55-10:40 am, 10:50-11:35 am and 11:45-12:30 am. After lunch there will be a long break until 4 pm to give the participants the possibility to read their notes, to check the references and to get in contact with the other participants in order to discuss on difficult points of the subject and to do group work on tutorial problems.

In the afternoon we will have 4 interactive rounds of 30 minutes of exercises, questions, discussion and presentations, such that the participants can practice the theoretical methods. The schedule is 4-4:30 pm, 4:40-5:10 pm, 5:20-5:50 pm and 6-6:30 pm. Talks of the participants on own results in the subject of the seminar are also possible but should be an exception (e.g. after dinner).

As the seminar will start on Monday morning, arrival day at Bedlewo is on Sunday, November 20 (afternoon or evening). A short round (about 20 minutes) of self-introduction of all participants is scheduled to Sunday evening after dinner. An excursion to the near-by city of Poznan is planned on Wednesday afternoon. The seminar will end on Friday after lunch with departure afternoon or evening.

There is no seminar fee and the stay of approved participants at the Mathematical Research and Conference Center of the Institute of Mathematics of the Polish Academy of Sciences in Bedlewo is free (full board). See the poster of the seminar for details how to apply for participation (to `seminars@mfo.de`, deadline: 1 September 2022).

Schedule of topics:

- Monday morning: Basic results from differential topology, algebraic topology, Lie theory and Riemannian geometry

- Monday afternoon: Classical Lie groups and well-known examples of homogeneous spaces

- Tuesday morning: Compact Lie groups and their representation theory, homogeneous spaces and metrics

- Tuesday afternoon: Examples of representations, low-dimensional isomorphisms of groups and spaces

- Wednesday morning: Systematic construction of all homogeneous spaces, classification in low dimensions, results on positive curvature

- Wednesday afternoon: Excursion to Poznan
- Thursday morning: Surgery classification of manifolds, Sullivan theory, index theorems of Atiyah, Singer and Patodi
- Thursday afternoon: Exotic homogeneous spaces
- Friday morning: Rational homotopy type of homogeneous spaces, moduli spaces of metrics
- Friday afternoon: End of the seminar

3 Prerequisites

The participants should have solid basic knowledge (at least one semester of lectures) in two or more of the following four subjects, where we also cite some standard literature:

- Differential topology: e.g. smooth manifolds, regular values, tangent bundle, vector fields and flows; see [1], [8], [2], [9]
- Algebraic topology: e.g. long exact sequence for the homotopy groups of a fibration, Mayer-Vietoris-Sequence for singular homology, Poincaré duality for manifolds; see [1], [17]
- Lie theory: e.g. compact Lie groups and their maximal tori, root system, representations, characters and weights; see [3], [16], [18]
- Differential geometry: e.g. geodesics, Riemannian curvature tensor, Ricci and scalar curvature, space forms; see [4], [10]

Concerning the more advanced methods, we will give an overview on the relevant results in the seminar. Participants who would like to deepen their knowledge can find more details in the literature on characteristic classes and bordism theory [15], surgery theory [22], rational homotopy theory [6], and spaces and moduli spaces of Riemannian metrics [19].

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